JOURNAL OF FUNCTIONAL ANALYSIS 116, 111–132 (1993)

Fourier Series of Radial Functions in Several Variables

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Communicated by L. Gross

Received July 19, 1992

We prove that the spherical partial sums of the Fourier series of the indicator function of a ball inside the cube of width 2π converge at the center of the ball if and only if the dimension is strictly less than three. For more general radial functions in three dimensions we give a necessary and sufficient condition for convergence. Related questions of non-localization and convergence of Fourier transforms of radial functions in three dimensions are examined.

1a. INTRODUCTION

The subject of Fourier series has been instrumental in the development of mathematical analysis for more than a century [B, H]. The rigorous formulation of both the Riemann and Lebesgue integrals was due in large part to the need of an integration theory for the general functions which are encountered in Fourier analysis. The spectral theory of self adjoint operators in Hilbert space is one of the many generalizations of Fourier analysis which finds its way into modern treatments of functional analysis.

* Research supported by the National Science Foundation

† Research supported by the Pew Charitable Trust.

0022-1236/93 $5.00

580/116/1-8

Fourier series in one dimension admit a *localization property*: the behavior of the Fourier series at a point depends only on the behavior of the function in a neighborhood of that point. In particular, if the function is zero on an interval, then the Fourier series converges to zero on that interval. This fails to be true in three dimensions, and the failure can be seen very explicitly, through the elementary examples given below in Section 3.

Since 1964 the general subject of one-dimensional Fourier analysis has been essentially complete. In that year L. Carleson proved that the Fourier series of a continuous (or more generally a square-integrable) function on the interval $[-\pi, \pi]$ converges to the function at *almost every point*, in the sense of Lebesgue measure. This was extended by Hunt to functions of the class L^p, $p > 1$, complementing previous results on convergence in the pth mean. Many years earlier Kolmogorov had produced an example of a function of the class L^1 for which the Fourier series diverges *everywhere.*

The situation in two and more dimensions is still relatively undeveloped with regard to pointwise convergence. Of course we can always expect mean convergence in L^2, by general theorems on the completeness of the complex exponential functions $e^{i\langle \mathbf{n}, \mathbf{x}\rangle}$. Fefferman showed that even this result cannot be generalized to L^p for $p \neq 2$. A nice exposition of these ideas is contained in [C].

It is our purpose here to explore convergence phenomena which arise when we attempt the Fourier expansion of *spherically symmetric functions* in two and three dimensions. Generalizations to higher dimensions are also outlined. We were partially led to these by various Mathematica graphs, which we have included below. An initial heuristic discussion of some of these ideas is contained in the final section of [GP].

1b. Notations

$\mathbf{Z}^p$ denotes the p-dimensional integer lattice, whose points are written $(n_1, ..., n_p)$, where n_i are arbitrary integers.

$\mathbf{T}^p$ denotes the p-dimensional cube of width 2π, whose points are written $(x_1, ..., x_p)$, where $-\pi \leqslant x_i \leqslant \pi$.

The inner product of $\mathbf{x} \in \mathbf{T}^p$ and $\mathbf{n} \in \mathbf{Z}^p$ is defined as $\langle \mathbf{n}, \mathbf{x}\rangle = \sum_{j=1}^p n_j x_j$. The norm is defined by $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$. The *unit sphere* $\mathbf{S}^{p-1} := \{(x_1, ..., x_p) : |\mathbf{x}| = 1\}$.

The *Fourier coefficients* of an integrable complex valued function $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{T}^p$, are defined by

$$A_{\mathbf{n}} := \frac{1}{(2\pi)^p} \int_{\mathbf{T}^p} f(\mathbf{x})\, e^{-i\langle \mathbf{n}, \mathbf{x}\rangle}\, dx_1 \cdots dx_p$$

where $i=\sqrt{-1}$. The *Fourier series* of f is the multiple series

$$\sum_{\mathbf{n}\in \mathbf{Z}^p} A_{\mathbf{n}} e^{i\langle \mathbf{n}, \mathbf{x}\rangle}.$$

The pointwise convergence of the Fourier series is defined in terms of the *spherical partial sums*

$$\sum_{\mathbf{n}\,:\,|\mathbf{n}|\leqslant R} A_{\mathbf{n}} e^{i\langle \mathbf{n}, \mathbf{x}\rangle}$$

when $R\uparrow\infty$. (This definition of convergence is natural from the viewpoint of spectral theory since it corresponds to the order of the decreasing sequence of eigenvalues of the Laplace operator $\Delta=\sum_{i=1}^{p}\partial^2/\partial x_i^2$.)

A function $f(\mathbf{x})$, $\mathbf{x}\in\mathbf{T}^p$, is said to be *radial* if there exists a complex-valued function $F(r)$, $0\leqslant r<\infty$, so that $f(\mathbf{x})=F(|\mathbf{x}|)$ whenever $\mathbf{x}\in\mathbf{T}^p$.

In the remainder of this paper we investigate the pointwise convergence of the Fourier series of the simplest radial functions. We make no use of theorems from Lebesgue integration or functional analysis. Only basic methods of classical analysis are used.

2. Radial Fourier Series in Two Variables

We consider the radial function $f\colon \mathbf{R}^2\to\mathbf{R}^1$ defined by

$$f(\mathbf{x})=1 \text{ for } 0\leqslant|\mathbf{x}|<a, \qquad f(\mathbf{x})=0 \text{ otherwise.}$$

Here $0<a\leqslant\pi$, so that we are looking at the indicator function of a disc which is inside (possibly touching the edges of) the basic square.

The Fourier coefficients are computed easily by evaluating the integrals in a system of polar coordinates r, θ as follows:

$$\begin{aligned} A_{\mathbf{n}} &= \frac{1}{(2\pi)^2}\int_{|\mathbf{x}|\leqslant a} e^{-i\langle \mathbf{n}, \mathbf{x}\rangle}\,dx_1\,dx_2 \\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_0^a e^{-i|\mathbf{n}|\,r\cos\theta}\,r\,dr\,d\theta, \end{aligned}$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{x}$.

For $|\mathbf{n}|=0$ the integral is easily seen to be $A_0=a^2/4\pi$. For $|\mathbf{n}|\neq0$ we may use the integral representation of the Bessel function J_0 [P1, p. 178] to write

$$A_{\mathbf{n}}=\frac{1}{2\pi}\int_0^a J_0(|\mathbf{n}|\,r)\,r\,dr.$$

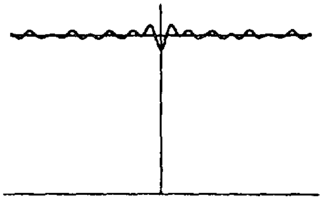

FIGURE 1

The latter integral is evaluated by using the differential equation satisfied by the Bessel function $G(r) := J_0(|\mathbf{n}|\, r)$, namely $G'' + (1/r)\, G' + |\mathbf{n}|^2\, G = 0$. If we multiply by r and integrate the resulting equation on the interval $[0, a]$, the equation $aG'(a) + |\mathbf{n}|^2 \int_r^a rG(r)\, dr = 0$ results. But the derivative is computed as $G'(r) = -|\mathbf{n}|\, J_1(|\mathbf{n}|\, r)$ which leads to the evaluation

$$A_{\mathbf{n}} = \frac{1}{2\, |\mathbf{n}|\, \pi}\, aJ_1(|\mathbf{n}|\, a), \qquad |\mathbf{n}| \neq 0.$$

The partial sum of this series for $a = \pi$, $|\mathbf{n}| \leqslant 20$, and $(x_1, x_2) = (x, 0)$ has been graphed using Mathematica and is depicted above (Fig. 1). It should be noted that there is a marked increase in the oscillation of the partial sum near $x = 0$. This is related to a "slower rate of convergence" of the series at this point.

The detailed analysis of this phenomenon is somewhat complicated, due to the appearance of the Bessel functions in the Fourier coefficients. The rigorous proof of convergence to the value 1 at $x = 0$ was carried out by Hardy and Landau [HL], who gave both a "real-variable" proof and a "complex-variable" proof. Rather than pursue these highly technical matters, we pass to a corresponding problem in three dimensions, where the computation of the Fourier coefficients involves only elementary functions.

3. Radial Fourier Series in Three Variables

In this section we consider the Fourier series of the radial function $f\colon \mathbf{R}^3 \to \mathbf{R}^1$ defined by

$$f(\mathbf{x}) = 1 \text{ for } 0 \leqslant |\mathbf{x}| < a, \qquad f(\mathbf{x}) = 0 \text{ otherwise}.$$

Here $0 < a \leqslant \pi$, so that we are looking at the indicator function of a ball which is inside (possibly touching the edges of) the basic cube.

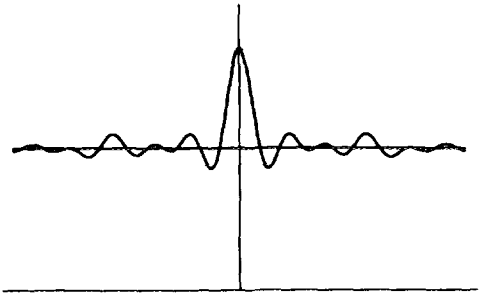

FIGURE 2

The Fourier coefficients are computed easily by evaluating the integrals in spherical coordinates r, ϕ, θ as follows:

$$\begin{aligned} A_{\mathbf{n}} &= \frac{1}{(2\pi)^3} \int_{|\mathbf{x}| \leqslant a} e^{-i\langle \mathbf{n}, \mathbf{x} \rangle}\, dx_1\, dx_2\, dx_3 \\ &= \frac{1}{(2\pi)^3} \int_0^a \int_0^\pi \int_{-\pi}^{\pi} e^{-i|\mathbf{n}| r \cos\phi} r^2 \sin\phi\, d\theta\, d\phi\, dr. \end{aligned}$$

For $|\mathbf{n}| = 0$ the integral is easily seen to be $A_0 = a^3/6\pi^2$. For $|\mathbf{n}| \neq 0$ we may evaluate the integral by elementary calculus to obtain

$$\begin{aligned} A_{\mathbf{n}} &= \frac{1}{2\pi^2 |\mathbf{n}|} \int_0^a \sin(|\mathbf{n}|\, r) r\, dr \\ &= \frac{1}{2\pi^2 |\mathbf{n}|} \left(-\frac{a \cos |\mathbf{n}|\, a}{|\mathbf{n}|} + \frac{\sin |\mathbf{n}|\, a}{|\mathbf{n}|^2} \right). \end{aligned}$$

The partial sum of this series for $a = \pi$, $|\mathbf{n}| \leqslant \sqrt{134}$ and $(x_1, x_2, x_3) = (x, 0, 0)$ has been graphed using Mathematica and is depicted in Fig. 2. The partial sum for $|\mathbf{n}| \leqslant \sqrt{155}$ is depicted in Fig. 3.

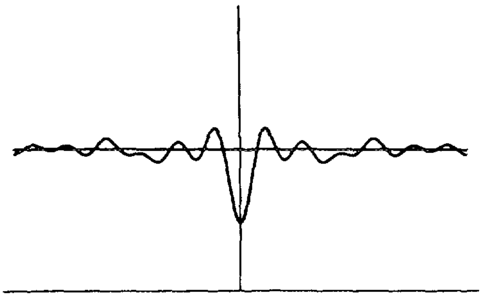

FIGURE 3

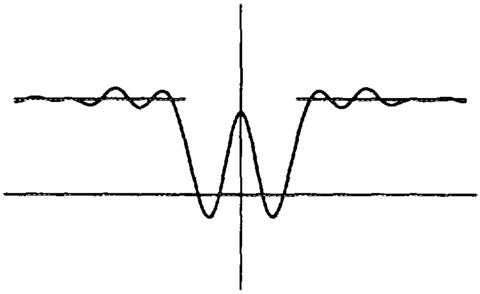

FIGURE 4

In contrast with the two-dimensional case, we note here a dramatic oscillation in the neighborhood of $x=0$, reminiscent of the Gibbs phenomenon from one-dimensional Fourier series. But in this case we have *divergence of the partial sums by oscillation*, which is totally unknown for one-dimensional Fourier series. The oscillation displayed by Figs. 2 and 3 illustrates that some of the partial sums go above the line $y=1$ while others go below. We prove in Section 5 that the Fourier series diverges at $\mathbf{x}=\mathbf{0}$.

This example may be modified to produce a simple example of the failure of localization in three dimensions. We consider the indicator function of an annulus:

$$f(\mathbf{x})=1 \text{ for } a \leqslant |\mathbf{x}| \leqslant b \qquad \text{and} \qquad f(\mathbf{x})=0 \text{ otherwise.}$$

Here $0<a<b\leqslant\pi$. The function is zero in a neighborhood of $\mathbf{x}=\mathbf{0}$ but the Fourier series diverges by oscillation at $\mathbf{x}=\mathbf{0}$. Figures 4 and 5 are graphs of the partial sums of this series with $(x_1, x_2, x_3)=(x, 0, 0)$. In Fig. 4, $a=\pi/4$, $b=\pi$, and $|\mathbf{n}|\leqslant\sqrt{134}$; in Fig. 5, $a=1$, $b=3$, and $|\mathbf{n}|\leqslant\sqrt{125}$. We see a phenomenon similar to the one displayed in Figs. 2 and 3; some of the partial sums are positive at $x=0$ and others are negative.

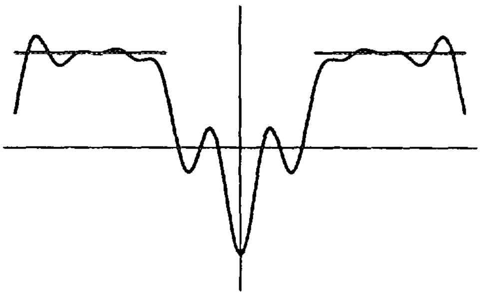

FIGURE 5

Previous examples of the failure of localization in low dimensions have depended on limiting procedures, such as infinite series, the principle of uniform boundedness, or other non-constructive methods of functional analysis. We refer to the article of Ash [A] and the references cited therein.

4. The Case of Higher Dimensions

It is natural to extend the previous discussion to Fourier series in higher dimensions. In this section we find an asymptotic formula for the Fourier coefficient $A_{\mathbf{n}}$, $|\mathbf{n}| \uparrow \infty$. In the next section we use this to give the rigorous analysis of the resulting Fourier series.

The Fourier coefficients of the indicator function of the ball $B = \{\mathbf{x} : |\mathbf{x}| \leqslant a\}$ are

$$A_{\mathbf{n}} = \frac{1}{(2\pi)^p} \int_B e^{-i\langle \mathbf{n}, \mathbf{x} \rangle}\, d\mathbf{x}$$

$$= \frac{1}{(2\pi)^p} \int_B \cos\langle \mathbf{n}, \mathbf{x} \rangle\, d\mathbf{x}.$$

As before, here $0 < a \leqslant \pi$, so that we are looking at the indicator function of a ball which is inside (possibly touching the edges of) the basic cube. Choosing the x_1-axis in the direction of $\mathbf{n}$ and letting $\mathbf{x}'$ denote the remaining coordinates, we have, for $\mathbf{n} \neq \mathbf{0}$:

$$A_{\mathbf{n}} = \frac{1}{(2\pi)^p} \int_{|\mathbf{x}'| \leqslant a} \int_{-\sqrt{a^2 - |\mathbf{x}'|^2}}^{\sqrt{a^2 - |\mathbf{x}'|^2}} \cos(|\mathbf{n}|\, x_1)\, dx_1\, d\mathbf{x}'$$

$$= \frac{2}{|\mathbf{n}|\, (2\pi)^p} \int_{|\mathbf{x}'| \leqslant a} \sin(|\mathbf{n}| \sqrt{a^2 - |\mathbf{x}'|^2})\, d\mathbf{x}'.$$

Let $a\mathbf{y}' = \mathbf{x}'$, $\mathbf{y}' = r\omega$, $\omega \in \mathbf{S}^{p-2}$. Thus

$$A_{\mathbf{n}} = \frac{2a^{p-1}}{|\mathbf{n}|\, (2\pi)^p} \int_{|\mathbf{y}'| < 1} \sin(|\mathbf{n}|\, a \sqrt{1 - |\mathbf{y}'|^2})\, d\mathbf{y}'$$

$$= \frac{2a^{p-1}}{|\mathbf{n}|\, (2\pi)^p} \int_{\mathbf{S}^{p-2}} \int_0^1 \sin(|\mathbf{n}|\, a \sqrt{1 - r^2})\, r^{p-2}\, dr\, d\omega$$

$$= \frac{2a^{p-1}\omega_{p-2}}{|\mathbf{n}|\, (2\pi)^p} \int_0^1 \sin(|\mathbf{n}|\, a \sqrt{1 - r^2})\, r^{p-2}\, dr.$$

Here we have used the notation $\omega_{p-2} = \mathrm{Vol}(\mathbf{S}^{p-2}) = (2\pi^{(p-1)/2})/\Gamma((p-1)/2))$. Letting $u = \sqrt{1-r^2}$, we have $du = -r\,dr/\sqrt{1-r^2}$, $r\,dr = -u\,du$, and

$$A_{\mathbf{n}} = \left(\frac{C_p}{|\mathbf{n}|}\right) \int_{-1}^{1} \sin(|\mathbf{n}|\,au)(1-u^2)^{(p-3)/2}\,u\,du$$

where we have noted that the final integrand is an even function to extend the integration to the interval $-1 < u < 1$. Here and below we use the notation C_p to denote a positive constant which may also depend upon a.

If $p \geqslant 3$ is odd then the integrand is a polynomial times a trigonometric function and the integral can be evaluated by successive integration-by-parts. To do this, let

$$I(k) = \int_{-1}^{1} \sin(bx)(1-x^2)^k\,x\,dx, \qquad k = 0, 1, 2, ..., \quad b \in \mathbf{R},$$

and

$$J(k) = \int_{-1}^{1} \cos(bx)(1-x^2)^k\,dx, \qquad k = 0, 1, 2, ..., \quad b \in \mathbf{R}.$$

Then for k a positive integer, integrating $I(k)$ by parts gives

$$I(k) = \frac{2k+1}{b}\,J(k) - \frac{2k}{b}\,J(k-1).$$

Integrating $J(k)$ by parts gives

$$J(k) = \frac{2k}{b}\,I(k-1)$$

so

$$I(k) = \frac{2k(2k+1)}{b^2}\,I(k-1) - \frac{4k(k-1)}{b^2}\,I(k-2).$$

Taking $k = (p-3)/2$, after $(p-1)/2$ similar integrations, we find the principal contribution and obtain the asymptotic formula

$$A_{\mathbf{n}} = \begin{cases} \dfrac{C_p}{|\mathbf{n}|^{(p+1)/2}} \left[\cos(|\mathbf{n}|\,a) + O\left(\dfrac{1}{|\mathbf{n}|}\right)\right], & \dfrac{p-1}{2} \text{ odd} \\[2ex] \dfrac{C_p}{|\mathbf{n}|^{(p+1)/2}} \left[\sin(|\mathbf{n}|\,a) + O\left(\dfrac{1}{|\mathbf{n}|}\right)\right], & \dfrac{p-1}{2} \text{ even.} \end{cases}$$

This generalizes the calculation in the previous section for the case $p=3$.

If $p \geqslant 2$ is even we transform the integral for $A_{\mathbf{n}}$ by writing $u=\cos\theta$, $t=|\mathbf{n}|$, to obtain

$$|\mathbf{n}|\, A_{\mathbf{n}} = C_p \int_0^\pi \sin^{p-2}\theta \cos\theta \sin(at\cos\theta)\, d\theta$$

$$= C_p \operatorname{Im} \int_0^\pi \sin^{p-2}\theta \cos\theta e^{iat\cos\theta}\, d\theta.$$

The latter integral can be integrated by parts $(p-2)/2$ times. Each time the endpoint term vanishes and we obtain

$$\operatorname{Im} \frac{C_p}{(ita)^{(p-2)/2}} \int_0^\pi F(\theta)\, e^{iat\cos\theta}\, d\theta,$$

where $F(\theta)$ is an explicit trigonometric polynomial.

This integral can be evaluated by the method of stationary phase [P1, Sect. 6.5], which shows in this case that

$$\int_0^\pi F(\theta)\, e^{iat\cos\theta}\, d\theta$$

$$= \sqrt{\frac{\pi}{2at}} \left[F(0)\, e^{iat} e^{-i\pi/4} + F(\pi)\, e^{-iat} e^{i\pi/4} + O\left(\frac{1}{\sqrt{t}}\right)\right], \qquad t \uparrow \infty.$$

The trigonometric polynomial satisfies $F(0) = |1 \times 3 \times 5 \cdots \times (p-3)|$ and $F(\pi) = \pm|F(0)|$, where the sign is positive if $p=4, 8, 12, \ldots$ and the sign is negative if $p=2, 6, 10, \ldots$. This leads to the asymptotic form for p even as

$$A_{\mathbf{n}} = \begin{cases} \dfrac{C_p}{|\mathbf{n}|^{(p+1)/2}} \left[\sin\left(|\mathbf{n}|\, a - \dfrac{\pi}{4}\right) + O\left(\dfrac{1}{\sqrt{|\mathbf{n}|}}\right)\right], & \dfrac{p}{2} \text{ odd} \\[2ex] \dfrac{C_p}{|\mathbf{n}|^{(p+1)/2}} \left[\cos\left(|\mathbf{n}|\, a - \dfrac{\pi}{4}\right) + O\left(\dfrac{1}{\sqrt{|\mathbf{n}|}}\right)\right], & \dfrac{p}{2} \text{ even.} \end{cases}$$

A similar calculation is valid for the integral which is obtained by formal differentiation with respect to t. These calculations never use the fact that $\mathbf{n} \in \mathbf{Z}^p$, so they hold for

$$A_\xi := \frac{1}{(2\pi)^p} \int_B e^{-i\langle \xi, \mathbf{x}\rangle}\, d\mathbf{x}, \qquad \xi \in \mathbf{R}^p,$$

as well. We can summarize the above calculations for all $p \geqslant 2$ as follows.

PROPOSITION. *There exists a smooth function* $A(t)$, $0<t<\infty$, *so that the Fourier coefficients* $A_{\mathbf{n}}$ *and the function* A_ξ *are given by* $A_{\mathbf{n}} = A(|\mathbf{n}|)$, $\mathbf{n} \in \mathbf{Z}^p$, $A_\xi = A(|\xi|)$, $\xi \in \mathbf{R}^p$,

$$A(t) = \frac{C_p}{t^{(p+1)/2}}\left[\sin\left(at - \frac{(p-1)\pi}{4}\right) + O\left(\frac{1}{\sqrt{t}}\right)\right], \qquad t \uparrow \infty,$$

and

$$A'(t) = \frac{aC_p}{t^{(p+1)/2}}\left[\cos\left(at - \frac{(p-1)\pi}{4}\right) + O\left(\frac{1}{\sqrt{t}}\right)\right], \qquad t \uparrow \infty,$$

where $C_p = C_p(a) > 0$.

5. Rigorous Analysis of the Fourier Series at $\mathbf{x} = \mathbf{0}$

Having obtained a formula for the Fourier coefficients, we are now in a position to give a rigorous proof of the divergence of the partial sums in dimensions $p \geqslant 3$. This is accomplished by means of an asymptotic formula for the number of lattice points within a sphere, obtained by Landau [L]. The statement is that if we define $N(\lambda) = \#\{\mathbf{n} \in Z^p : |\mathbf{n}| \leqslant \lambda\}$, then there is a constant c_p depending only on the dimension, so that

$$N(\lambda) = c_p \lambda^p + O(\lambda^{p(p-1)/(p+1)}), \qquad \lambda \uparrow \infty.$$

For example, in the case of three dimensions, we have $N(\lambda) = 4\pi\lambda^3/3 + O(\lambda^{3/2})$. We also use the notation $n(\lambda) = \#\{\mathbf{n} \in \mathbf{Z}^p : |\mathbf{n}| = \lambda\}$, so that

$$N(\lambda) = \sum_{r \leqslant \lambda} n(r).$$

Modern treatments of Landau's estimate are in Guillemin [G], with a proof that also gives an estimate due to Hlawka and Herz for the number of lattice points inside a strictly convex domain, and in Lax and Phillips [LP], with a proof that also holds for other lattices. We include a proof in the Appendix.

Landau's estimate is now used to prove the following:

THEOREM. *If* $0 < a \leqslant \pi$, $p \geqslant 3$, *then the p-dimensional Fourier series of the indicator function of the ball* $|\mathbf{x}| \leqslant a$ *diverges at the center* $\mathbf{x} = \mathbf{0}$.

Proof. The Fourier coefficient $A_{\mathbf{n}}$ is itself a radial function $A_{\mathbf{n}} = A(|\mathbf{n}|)$ as we have found above. The function $A(r)$ is naturally defined and differentiable for *all* $r > 0$. We organize the partial sum of the Fourier series according to a radial/angular decomposition. At the center of the ball the partial sum of the Fourier series is

$$\begin{aligned}\sum_{\mathbf{n}\,:\,|\mathbf{n}|\leqslant R} A_{\mathbf{n}} &= \sum_{r\,:\,0\leqslant r\leqslant R}\ \sum_{\mathbf{n}\,:\,|\mathbf{n}|=r} A(|\mathbf{n}|)\\ &= \sum_{r\,:\,0\leqslant r\leqslant R} n(r)\,A(r)\\ &= \int_0^R A(r)\,dN(r) + A_0,\end{aligned}$$

where the latter integral is a Stieltjes integral. This may be integrated by parts to obtain

$$\sum_{\mathbf{n}\,:\,|\mathbf{n}|\leqslant R} A_{\mathbf{n}} = A(R)\,N(R) - \int_0^R A'(r)\,N(r)\,dr.$$

This is to be compared to the integral

$$\int_0^R A(r)\,pc_p r^{p-1}\,dr = A(R)\,c_p R^p - \int_0^R A'(r)\,c_p r^p\,dr.$$

If we subtract the second from the first, we obtain

$$\begin{aligned}\sum_{\mathbf{n}\,:\,|\mathbf{n}|\leqslant R} A_{\mathbf{n}} - \int_0^R A(r)\,pc_p r^{p-1}\,dr &= A(R)[N(R) - c_p R^p]\\ &\quad - \int_0^R A'(r)[N(r) - c_p r^p]\,dr.\end{aligned}$$

We now apply this to the sum obtained between the consecutive zeros x_k of the trigonometric function $t \to \sin(at - (p-1)\pi/4)$ which occurs in the asymptotic estimate for $A(t)$. Thus

$$\begin{aligned}\sum_{\mathbf{n}\,:\,x_{k-1}<|\mathbf{n}|\leqslant x_k} A_{\mathbf{n}} - \int_{x_{k-1}}^{x_k} A(r)\,pc_p r^{p-1}\,dr &= A(r)[N(r) - c_p r^p]\,\Big|_{x_{k-1}}^{x_k}\\ &\quad - \int_{x_{k-1}}^{x_k} A'(r)[N(r) - c_p r^p]\,dr.\end{aligned}$$

Both terms on the right may be estimated using Landau's asymptotic formula, to obtain

$$\sum_{\mathbf{n}\,:\,x_{k-1}<|\mathbf{n}|\leqslant x_k} A_{\mathbf{n}} - \int_{x_{k-1}}^{x_k} A(r)\, pc_p r^{p-1}\, dr = O(k^{(p-3)/2-(p-1)/(p+1)}), \quad k \uparrow \infty.$$

But the integral $\int_{x_{k-1}}^{x_k} A(r)\, pc_p r^{p-1}\, dr$ has the explicit asymptotic behavior

$$C_p k^{(p-3)/2}\left[\cos\left(ax_k - \frac{(p-1)\pi}{4}\right) + O\left(\frac{1}{\sqrt{k}}\right)\right], \qquad k \uparrow \infty.$$

Hence this term is the predominant contribution and

$$\sum_{\mathbf{n}\,:\,x_{k-1}<|\mathbf{n}|\leqslant x_k} A_{\mathbf{n}} = C_p k^{(p-3)/2} \cos\left(ax_k - \frac{(p-1)\pi}{4}\right) + O(k^{(p-3)/2-1/2})$$

when $k \uparrow \infty$. To prove the asserted divergence, we consider separately the cases $p=3$ and $p>3$. In the first case, convergence of the series implies, by the Cauchy criterion, that the sum $S_k := \sum_{\mathbf{n}\,:\,x_{k-1}<|\mathbf{n}|\leqslant x_k} A_{\mathbf{n}}$ tends to zero when $k \uparrow \infty$. But the remainder term is $O(k^{-1/2})$ which tends to zero, while the leading terms has the above-displayed oscillatory behavior when $k \uparrow \infty$. Therefore we conclude divergence in this case.

In case $p>3$ we claim that the terms of the original series diverge unboundedly. Indeed, if this were not the case, then the sums S_k would remain bounded, so $k^{-(p-3)/2} S_k$ would tend to zero. However, when we divide the estimate for S_k by $k^{(p-3)/2}$ the remainder term tends to zero, but the leading terms does not.

This concludes the proof that the original partial sums are divergent for every $p \geqslant 3$.

6. A Positive Result in Three Dimensions

Theorem. *Let $f(x_1, x_2, x_3) = F(r)$ for $0 \leqslant r \leqslant a$ and $f(x_1, x_2, x_3) = 0$ elsewhere, where $0 < a \leqslant \pi$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and F is a smooth function on the interval $[0, a]$. If $F(a) = 0$ then the Fourier series converges for all $\mathbf{x} \in \mathbf{T}^3$. Conversely, if the Fourier series converges at $\mathbf{x} = \mathbf{0}$ then $F(a) = 0$.*

This will be proved by developing the following asymptotic behavior of the Fourier coefficients.

LEMMA. *Let* $f(x_1, x_2, x_3) = F(r)$ *for* $0 \leqslant r \leqslant a$ *and* $f(x_1, x_2, x_3) = 0$ *elsewhere, where* $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ *and* F *is a smooth function on the interval* $[0, a]$. *Then the Fourier coefficients depend only on* $|\mathbf{n}|$ *and satisfy*

$$A_{\mathbf{n}} = -\frac{a \cos |\mathbf{n}|\, a}{2\pi^2 |\mathbf{n}|^2} F(a) + \frac{\sin |\mathbf{n}|\, a}{2\pi^2 |\mathbf{n}|^3} \frac{d}{dr} [rF(r)](a-0)$$

$$+ O\left(\frac{1}{|\mathbf{n}|^4}\right), \qquad |\mathbf{n}| \uparrow \infty.$$

Proof. We can evaluate the Fourier coefficients by taking a system of spherical coordinates (r, θ, ϕ), where ϕ is the angle which the radius vector (x_1, x_2, x_3) makes with the vector (n_1, n_2, n_3). Then we have, for $\mathbf{n} \neq \mathbf{0}$,

$$\begin{aligned}
A_{\mathbf{n}} &= \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x_1, x_2, x_3)\, e^{-i(n_1 x_1 + n_2 x_2 + n_3 x_3)}\, dx_1\, dx_2\, dx_3 \\
&= \frac{1}{(2\pi)^3} \int_0^a \int_0^{\pi} \int_0^{2\pi} F(r)\, e^{-i|\mathbf{n}| r \cos\phi}\, r^2 \sin\phi\, d\theta\, d\phi\, dr \\
&= \frac{1}{2\pi^2} \int_0^a F(r) \frac{\sin |\mathbf{n}|\, r}{|\mathbf{n}|\, r} r^2\, dr \\
&= \frac{1}{2\pi^2 |\mathbf{n}|} \int_0^a rF(r) \sin |\mathbf{n}|\, r\, dr \\
&= -\frac{1}{2\pi^2 |\mathbf{n}|^2} \int_0^a rF(r) \frac{d}{dr} (\cos |\mathbf{n}|\, r)\, dr \\
&= -\frac{1}{2\pi^2 |\mathbf{n}|^2} \left[aF(a) \cos |\mathbf{n}|\, a - \int_0^a \cos |\mathbf{n}|\, r \frac{d}{dr} [rF(r)]\, dr \right].
\end{aligned}$$

The final integral can be integrated-by-parts once again to obtain the indicated result, with the indicated remainder term, proving the lemma.

Therefore the Fourier series at the center $\mathbf{x} = \mathbf{0}$ has the form

$$\sum_{\mathbf{n} \in \mathbf{Z}^3} A_{\mathbf{n}} = A_{\mathbf{0}} + \sum_{\mathbf{n} \in \mathbf{Z}^3 \setminus \{\mathbf{0}\}} \left[\frac{\cos |\mathbf{n}|\, a}{|\mathbf{n}|^2} C_1 + \frac{\sin |\mathbf{n}|\, a}{|\mathbf{n}|^3} C_2 + O\left(\frac{1}{|\mathbf{n}|^4}\right) \right],$$

where the constants C_1, C_2 are given above.

In order to prove the theorem, we use Landau's asymptotic estimate mentioned above. We must compare a sum of the form

$$\textstyle\sum_{|\mathbf{n}| \leqslant R} A(|\mathbf{n}|) \qquad \text{with} \qquad \int_0^R A(r)\, 4\pi r^2\, dr.$$

To do this we integrate both by parts:

$$\sum_{|\mathbf{n}| \leq R} A(|\mathbf{n}|) = A(R)\, N(R) - \int_0^R A'(r)\, N(r)\, dr$$

$$\int_0^R A(r)\, 4\pi r^2\, dr = A(R)\,\frac{4\pi R^3}{3} - \int_0^R A'(r)\,\frac{4\pi r^3}{3}\, dr.$$

Now we subtract these two equations and analyze the two terms separately. Since $F(a) = 0$, the function $A(r)$ has the form $A(r) = C_2(\sin ar/r^3) + O(1/r^4)$ with $A'(r) = aC_2(\cos ar/r^3) + O(1/r^4)$, $r \uparrow \infty$.

Therefore $[N(R) - 4\pi R^3/3] \times A(R) = O(R^{3/2} \times R^{-3}) = O(R^{-3/2})$, $R \uparrow \infty$. Similarly, the difference of the two integrals is estimated by $\int_1^R r^{3/2} r^{-3}\, dr$, which converges absolutely as $R \uparrow \infty$. Therefore the difference

$$\sum_{|\mathbf{n}| \leq R} A(|\mathbf{n}|) - \int_0^R 4\pi r^2 A(r)\, dr$$

has a limit when $R \uparrow \infty$. But the integral is easily seen to be convergent, from the above form of $A(r)$. This proves that the Fourier series is convergent at $\mathbf{x} = \mathbf{0}$.

To prove convergence at $\mathbf{x} \neq \mathbf{0}$ we use Landau's estimate for trigonometric sums [L], proved in the Appendix: $N^{\mathbf{x}}(r) = O(r^{3/2})$, $r \to \infty$. The spherical partial sum is written

$$\sum_{|\mathbf{n}| \leq \mathbf{R}} A_{\mathbf{n}} e^{i\langle \mathbf{n}, \mathbf{x}\rangle} = \int_0^R A(r)\, dN^{\mathbf{x}}(r) + A_0$$

$$= A(R)\, N^{\mathbf{x}}(R) - \int_0^R A'(r)\, N^{\mathbf{x}}(r)\, dr.$$

Applying the appropriate estimate for $A'(r)$, we see that the last integral is absolutely convergent and the endpoint terms tends to zero. This completes the proof of convergence for $\mathbf{x} \neq \mathbf{0}$.

Conversely, if the Fourier series is convergent at $\mathbf{x} = \mathbf{0}$ then we may follow the above steps to see that the C_2 contribution produces a convergent series. If $C_1 \neq 0$ we conclude, as in the previous Section 5, that we must have divergence, a contradiction. Hence $F(a) = 0$.

7. An Overview with Fourier Transforms

As a concluding remark, we can gain further insight into the phenomena discussed here by means of the *Fourier transform* of radial functions. In this context some of the formulas become simpler, thus revealing more clearly the nature of the convergence/divergence phenomena.

The Fourier transform of an integrable function $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^p$, is defined as

$$\hat{f}(\mu) = \frac{1}{(2\pi)^p} \int_{\mathbf{R}^p} f(\mathbf{x})\, e^{-i\langle \mu, \mathbf{x} \rangle}\, d\mathbf{x}.$$

We attempt to retrieve the original function through the "spherical partial sum"

$$\lim_{R \uparrow \infty} \int_{|\mu| \leqslant R} \hat{f}(\mu)\, e^{i\langle \mu, \mathbf{x} \rangle}\, d\mu.$$

When we insert the definition of $\hat{f}(\mu)$ and interchange the order of integration, the latter integral is written as the convolution

$$\int_{\mathbf{R}^p} f(\mathbf{y})\, D_R(\mathbf{x} - \mathbf{y})\, d\mathbf{y}, \qquad D_R(\mathbf{y}) := \frac{1}{(2\pi)^p} \int_{|\mu| \leqslant R} e^{i\langle \mu, \mathbf{y} \rangle}\, d\mu = \tilde{D}_R(|\mathbf{y}|).$$

This is the "Dirichlet kernel" in the present context. It can be expressed explicitly in terms of Bessel functions, if necessary.

In case $p = 3$, by the calculation of Section 3, we have the elementary trigonometric function

$$\tilde{D}_R(z) = -\frac{1}{2\pi^2 z} \frac{d}{dz}\left(\frac{\sin Rz}{z}\right), \qquad z > 0.$$

When we apply this to a radial function $f(\mathbf{x}) = F(|\mathbf{x}|)$ and evaluate at $\mathbf{x} = 0$ we obtain the integral

$$\int_{\mathbf{R}^p} f(\mathbf{y})\, D_R(-\mathbf{y})\, d\mathbf{y} = -\frac{2}{\pi} \int_{0+}^{\infty} F(z)\, z^{-1} \frac{d}{dz}\left(\frac{\sin Rz}{z}\right) z^2\, dz.$$

Assuming that $F(z) = 0$ for $z > Z$ and is smooth for $0 \leqslant z \leqslant Z$, we can integrate-by-parts to obtain

$$\frac{2}{\pi}\left[-F(Z) \sin RZ + \int_{0+}^{Z} \frac{\sin Rz}{z} \frac{d}{dz}(zF(z))\, dz\right].$$

The final integral can be analyzed by the methods of one-dimensional Fourier analysis of piecewise smooth functions [P1, Sect. 1.2]. This gives

$$\lim_{R \uparrow \infty} \int_{0+}^{Z} \frac{\sin Rz}{z} \frac{d}{dz}(zF(z))\, dz = \frac{\pi}{2} F(0+).$$

Therefore the convergence of the spherical partial sum depends on the behavior of the term at the boundary $z=Z$. Clearly this will be zero if $F(Z)=0$; otherwise this term will oscillate when $R\uparrow\infty$, causing the partial sums to diverge. This agrees with the result of Section 6 for radial Fourier series in three dimensions. In the present context we can see this without directly estimating the Fourier transform, since we have the explicit form of the Dirichlet kernel D_R.

In case $p=2$ the calculation in Section 2 shows that the Dirichlet kernel is

$$\begin{aligned}D_R(\mathbf{y}) &= \frac{1}{(2\pi)^2}\int_0^R\int_0^{2\pi} e^{i\mu|\mathbf{y}|\cos\theta}\mu\,d\mu\,d\theta\\ &=\frac{1}{2\pi}\int_0^R J_0(\mu|\mathbf{y}|)\,\mu\,d\mu\\ &=\frac{R}{2\pi|\mathbf{y}|}J_1(R|\mathbf{y}|).\end{aligned}$$

Therefore the spherical partial sum of the Fourier transform of a radial function $f=F(r)$ at $\mathbf{x}=\mathbf{0}$ is

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} F(|\mathbf{y}|)\,R\,\frac{J_1(R|\mathbf{y}|)}{|\mathbf{y}|}\,dy_1\,dy_2.$$

In case $F(z)=0$ for $z>Z$, this may be written as the one-dimensional integral

$$\int_0^Z F(z)\,RJ_1(Rz)\,dz.$$

To prove convergence, we may assume that $F(0+)=0$ (otherwise replace $F(z)$ by $F(z)-F(0+)$ and recall that the improper integral $\int_0^\infty J_1(x)\,dx = -\int_0^\infty J_0'(x)\,dx=1$ [P1, pp. 180, 314]). Now

$$\begin{aligned}\int_0^Z F(z)\,RJ_1(Rz)\,dz &= -\int_0^Z F(z)\frac{d}{dz}J_0(Rz)\,dz\\ &= -J_0(RZ)\,F(Z)+\int_0^Z J_0(Rz)\,F'(z)\,dz.\end{aligned}$$

The boundary term tends to zero when $R\uparrow\infty$. To handle the integral, we note that it tends to zero whenever $F'(z)=z^n$ for some $n=0, 1, 2, \ldots$. But the continuous function F' can be uniformly approximated by a polynomial on the interval $0\leqslant z\leqslant Z$. Therefore the integral tends to zero for any C^1 function $F(z)$, $0\leqslant z\leqslant Z$.

We summarize as follows:

THEOREM. *Suppose that* $f(\mathbf{x}) = F(|\mathbf{x}|)$ *where* $F(z)$, $0 \leqslant z \leqslant Z$, *has a continuous derivative, and* $f\colon \mathbf{R}^p \to \mathbf{R}$ *is zero elsewhere. If* $p = 2$ *the spherical partial sum of the Fourier transform converges at* $\mathbf{x} = \mathbf{0}$ *to* $f(\mathbf{0})$. *If* $p = 3$ *this convergence takes place if and only if* $F(Z) = 0$.

Further resu'ts on the pointwise inversion of the multi-dimensional Fourier transform may be found in [P2].

APPENDIX: LANDAU'S ESTIMATE FOR TRIGONOMETRIC SUMS

In this section we give a self-contained development of Landau's estimates for the number of lattice points and for related trigonometric sums in $\mathbf{R}^p$, $p \geqslant 2$.

THEOREM (Landau [L]). *Let* $\alpha \in \mathbf{R}^p$, *and let*

$$N^\alpha(\lambda) := \sum_{\nu \in \mathbf{Z}^p,\ |\nu| \leqslant \lambda} e^{2\pi i \langle \alpha, \nu \rangle}$$

for $\lambda > 0$. *Then as* $\lambda \to \infty$, *for* $\alpha \in \mathbf{Z}^p$,

$$N^\alpha(\lambda) = N(\lambda) = c_p \lambda^p + O(\lambda^{p-2+2/(p+1)})$$

and, for $\alpha \notin \mathbf{Z}^p$,

$$N^\alpha(\lambda) = O(\lambda^{p-2+2/(p+1)}).$$

For the case of the lattice point estimate, $\alpha = 0$, our proof is essentially the proof in Guillemin [G], specialized to the case of the ball. We then modify this proof to obtain the case of trigonometric sums, $\alpha \notin \mathbf{Z}^p$.

We begin with the Poisson summation formula $\sum_{\nu \in \mathbf{Z}^p} f(\nu) = \sum_{\mu \in \mathbf{Z}^p} \hat{f}(\mu)$, where $f(x)$ is a smooth rapidly decreasing function on $\mathbf{R}^p$ and $\hat{f}(\mu) = \int_{\mathbf{R}^p} e^{-2\pi i \langle \mu, x \rangle} f(x)\, dx$ is the Fourier transform of the function $f(x)$.

The lattice sum $N^\alpha(\lambda)$ can be written as $\sum_{\nu \in \mathbf{Z}^p} f^\alpha_\lambda(\nu)$, where $f^\alpha_\lambda(x) = e^{2\pi i \langle \alpha, x \rangle} I_\lambda(x)$ and I_λ is the indicator function of the ball $\{|x| \leqslant \lambda\}$. In order to work with a smooth rapidly decreasing function, we regularize. Let $\rho(x)$ be an even, non-negative smooth function on $\mathbf{R}^p$ with $\int_{\mathbf{R}^p} \rho(x)\, dx = 1$ and $\rho(x) = 0$ for $|x| > 1$. The scaled function ρ_ε is $\rho_\varepsilon(x) := \varepsilon^{-p} \rho(x/\varepsilon)$.

The desired lattice sum is

$$N^\alpha(\lambda) = \sum_{\nu \in \mathbf{Z}^p,\ |\nu| \leqslant \lambda} e^{2\pi i \langle \alpha, \nu \rangle} = \sum_{\nu \in \mathbf{Z}^p} f^\alpha_\lambda(\nu)$$

and the regularized lattice sum is then defined as

$$N_\varepsilon^\alpha(\lambda) := \sum_{\nu \in \mathbf{Z}^p} (f_\lambda^\alpha * \rho_\varepsilon)(\nu).$$

LEMMA 1. *Let*

$$b(\varepsilon, \alpha) := \int_{|z| \leqslant \varepsilon} [e^{-2\pi i \langle \alpha, z \rangle} - 1] \, \rho_\varepsilon(z) \, dz.$$

Then

$$|N_\varepsilon^\alpha(\lambda) - N^\alpha(\lambda) - b(\varepsilon, \alpha) \, N^\alpha(\lambda - \varepsilon)| \leqslant 2[N^0(\lambda + \varepsilon) - N^0(\lambda - \varepsilon)],$$

and

$$|b(\varepsilon, \alpha)| \leqslant 2\pi^2 \varepsilon^2 \, |\alpha|^2.$$

Also,

$$N_\varepsilon^0(\lambda - \varepsilon) \leqslant N^0(\lambda) \leqslant N_\varepsilon^0(\lambda + \varepsilon).$$

Proof. From the above definitions, we have

$$\begin{aligned} N_\varepsilon^\alpha(\lambda) - N^\alpha(\lambda) &= \sum_{\nu \in \mathbf{Z}^p} [(f_\lambda^\alpha * \rho_\varepsilon)(\nu) - f_\lambda^\alpha(\nu)] \\ &= \left[\sum_{|\nu| \leqslant \lambda - \varepsilon} + \sum_{\lambda - \varepsilon < |\nu| \leqslant \lambda + \varepsilon} + \sum_{|\nu| > \lambda + \varepsilon} \right] [(f_\lambda^\alpha * \rho_\varepsilon)(\nu) - f_\lambda^\alpha(\nu)]. \end{aligned}$$

In the region $|\nu| > \lambda + \varepsilon$ we have

$$\begin{aligned} (f_\lambda^\alpha * \rho_\varepsilon)(\nu) &= \int_{|z| \leqslant \varepsilon} e^{2\pi i \langle \alpha, \nu - z \rangle} \rho_\varepsilon(z) \, I_\lambda(\nu - z) \, dz = 0, \\ f_\lambda^\alpha(\nu) &= I_\lambda(\nu) \, e^{2\pi i \langle \alpha, \nu \rangle} = 0. \end{aligned}$$

In the region $|\nu| \leqslant \lambda - \varepsilon$ we note that $|z| \leqslant \varepsilon$ implies that $|\nu - z| \leqslant \lambda - \varepsilon + \varepsilon = \lambda$, so that

$$\int_{|z| \leqslant \varepsilon} I_\lambda(\nu - z) \, e^{2\pi i \langle \alpha, \nu - z \rangle} \rho_\varepsilon(z) \, dz = \int_{|z| \leqslant \varepsilon} e^{2\pi i \langle \alpha, \nu - z \rangle} \rho_\varepsilon(z) \, dz,$$

$$\begin{aligned} (f_\lambda^\alpha * \rho_\varepsilon)(\nu) - f_\lambda^\alpha(\nu) &= \int_{|z| \leqslant \varepsilon} [e^{2\pi i \langle \alpha, \nu - z \rangle} - e^{2\pi i \langle \alpha, \nu \rangle}] \, \rho_\varepsilon(z) \, dz \\ &= e^{2\pi i \langle \alpha, \nu \rangle} b(\varepsilon, \alpha). \end{aligned}$$

Because ρ is even, we see that the integral $\int_{|z| \leqslant \varepsilon} \sin 2\pi\langle \alpha, z\rangle \, \rho_\varepsilon(z)\, dz = 0$. Hence, when we use the Taylor expansion inequality $|\cos\theta - 1| \leqslant \theta^2/2$ and the Schwarz inequality, we obtain the inequality $|b(\varepsilon, \alpha)| \leqslant 2\pi^2 \int_{|z| \leqslant \varepsilon} \langle \alpha, z\rangle^2 \rho_\varepsilon(z)\, dz \leqslant 2\pi^2 |\alpha|^2 \varepsilon^2$.

Next we estimate the sum for $\lambda - \varepsilon < |\nu| \leqslant \lambda + \varepsilon$. This is straightforward, once we notice that $|f_\lambda^\alpha * \rho_\varepsilon(\nu)|$ and $|f_\lambda^\alpha(\nu)|$ are both at most 1, so that

$$\left| \sum_{\lambda - \varepsilon < |\nu| \leqslant \lambda + \varepsilon} f_\lambda^\alpha * \rho_\varepsilon(\nu) - f_\lambda^\alpha(\nu) \right| \leqslant 2 \sum_{\lambda - \varepsilon < |\nu| \leqslant \lambda + \varepsilon} 1$$
$$= 2[N^0(\lambda + \varepsilon) - N^0(\lambda - \varepsilon)].$$

This proves the first inequality of the lemma. Finally, we note that if $\alpha = 0$, then $0 \leqslant f_\lambda^0$ so, by the definitions of f_λ^0 and ρ_ε,

$$(f_{\lambda - \varepsilon}^0 * \rho_\varepsilon)(\nu) \leqslant f_\lambda^0(\nu) \leqslant (f_{\lambda + \varepsilon}^0 * \rho_\varepsilon)(\nu).$$

The last inequality now follows immediately. This completes the proof of the lemma.

LEMMA 2. *Let* $\alpha = 0$ or $\alpha \in \mathbf{R}^p \backslash \mathbf{Z}^p$. *Then*

$$|N_\varepsilon^\alpha(\lambda) - c^\alpha(\lambda)| \leqslant c(p, \alpha) \left(\frac{\lambda}{\varepsilon}\right)^{(p-1)/2}$$

for all $\varepsilon < 1$ *and* $\lambda > 1$ *where*

$$c^\alpha(\lambda) = \begin{cases} c_p \lambda^p, & \alpha = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By the Poisson summation formula

$$N_\varepsilon^\alpha(\lambda) = \sum_{\mu \in \mathbf{Z}^p} \hat{f}_\lambda^\alpha(\mu)\, \hat{\rho}_\varepsilon(\mu).$$

Using the notation of Section 4 and taking $a = 1$, we see that $\hat{f}_\lambda^\alpha(\mu) = (2\pi\lambda)^p A_{2\pi\lambda(\mu - \alpha)}$. Hence, by the proposition of Section 4,

$$|\hat{f}_\lambda^\alpha(\mu)| \leqslant C_p \frac{\lambda^p}{(\lambda\, |\mu - \alpha|)^{(p+1)/2}}.$$

Also, $\hat{\rho}_\varepsilon(\mu) = \hat{\rho}(\varepsilon\mu)$ and, because ρ is smooth with compact support, for any integer N

$$|\hat{\rho}(\varepsilon\mu)| \leqslant c_N \left[\frac{1}{1+|\varepsilon\mu|^2}\right]^N.$$

For $\alpha \in \mathbf{R}^p \backslash \mathbf{Z}^p$, let $c(\alpha)$ denote the distance from α to $\mathbf{Z}^p$ and let $C(\alpha) = \min(1/2, c(\alpha)/2\,|\alpha|)$. By considering the cases $|\mu| \geqslant 2\,|\alpha|$ and $|\mu| < 2\,|\alpha|$, we see that for $\mu \in \mathbf{Z}^p$, $|\mu - \alpha|/|\mu| \geqslant C(\alpha) > 0$. Thus, for $N \geqslant p/2$,

$$\begin{aligned}
|N_\varepsilon^\alpha(\lambda) - \hat{f}_\lambda^\alpha(0)| &\leqslant \frac{c(N, \alpha, p)\,\lambda^p}{\lambda^{(p+1)/2}} \sum_{\mu \neq 0} \frac{1}{|\mu|^{(p+1)/2}} \left[\frac{1}{1+|\varepsilon\mu|^2}\right]^N \\
&\leqslant \frac{C(N, \alpha, p)\,\lambda^p}{\lambda^{(p+1)/2}} \int_{\mathbf{R}^p} \frac{1}{|\mu|^{(p+1)/2}} \left[\frac{1}{1+|\varepsilon\mu|^2}\right]^N d\mu \\
&= C(N, \alpha, p)\,\lambda^{(p-1)/2} \int_{\mathbf{R}^p} \frac{\varepsilon^{(p+1)/2}}{|y|^{(p+1)/2}} \left[\frac{1}{1+|y|^2}\right]^N \varepsilon^{-p}\,dy \\
&= c_{p,N,\alpha} \left(\frac{\lambda}{\varepsilon}\right)^{(p-1)/2}.
\end{aligned}$$

The lemma now follows from the facts that $\hat{f}_\lambda^0(0) = \mathrm{Vol}(\{|\xi| \leqslant \lambda\}) = c_p\lambda^p$ and that, for $\alpha \neq 0$, $\hat{f}_\lambda^\alpha(0) = (2\pi\lambda)^p A_{(-2\pi\lambda\alpha)} = O(\lambda^{(p-1)/2})$ as $\lambda \to \infty$.

We can now use these lemmas to prove Landau's estimate for $\alpha = 0$. Applying Lemma 2 with λ replaced by $\lambda \pm \varepsilon$ and using the last statement of Lemma 1, we see that

$$N^0(\lambda) = c_p\lambda^p + O(\varepsilon\lambda^{p-1}) + O\left(\frac{\lambda}{\varepsilon}\right)^{(p-1)/2} \qquad \text{as} \quad \lambda \to \infty,$$

where we take $\varepsilon = \lambda^{-1+2/(p+1)}$. Each of the error terms is $O(\lambda^{p-2+2/(p+1)})$. This proves Landau's estimate for $\alpha = 0$. The estimate for $\alpha \in \mathbf{Z}^p$ now follows because $N^\alpha = N^0$ in this case.

In case $\alpha \notin \mathbf{Z}^p$, we first take $\varepsilon = \lambda^{-1+2/(p+3)}$ and use Landau's estimate for N^0 to see that

$$\varepsilon\lambda^{p-1} = \lambda^{p-2+2/(p+3)}$$

$$\left(\frac{\lambda}{\varepsilon}\right)^{(p-1)/2} = \lambda^{(p-1)/2}\lambda^{(p-1)/2-(p-1)/(p+3)} = \lambda^{p-2+4/(p+3)}$$

$$\varepsilon^2\,|N^\alpha(\lambda - \varepsilon)| \leqslant \varepsilon^2 N^0(\lambda) = O(\lambda^{p-2+4/(p+3)}).$$

Therefore, by Lemma 2, $N_\varepsilon^\alpha(\lambda) = O(\lambda^{p-2+4/(p+3)})$. Landau's estimate for $\alpha = 0$ shows that

$$N^0(\lambda+\varepsilon) - N^0(\lambda-\varepsilon) = O(\varepsilon\lambda^{p-1}) + O(\lambda^{p-2+2/(p+1)}).$$

Hence, by Lemma 1, $N^\alpha(\lambda) = O(\lambda^{p-2+4/(p+3)})$. Now we estimate $N^\alpha(\lambda)$ again, using the choice $\varepsilon = \lambda^{-1+2/(p+1)}$. As in the proof for $\alpha = 0$, $\varepsilon\lambda^{p-1}$ and $(\lambda/\varepsilon)^{(p-1)/2}$ are $O(\lambda^{p-2+2/(p+1)})$. Our initial estimate for N^α gives

$$\begin{aligned} \varepsilon^2\,|N^\alpha(\lambda-\varepsilon)| &= O(\varepsilon^2(\lambda-\varepsilon)^{p-2+4/(p+3)}) \\ &= O(\varepsilon^2\lambda^{p-2+4/(p+3)}) \\ &= O(\lambda^{p-2+2/(p+1)}). \end{aligned}$$

Hence, by Lemma 2 and the first two inequalities of Lemma 1, $N^\alpha(\lambda) = O(\lambda^{p-2+2/(p+1)})$ whenever $p \geqslant 2$ and $\alpha \notin \mathbf{Z}^p$. This completes the proof.

Acknowledgments

We thank S. R. S. Varadhan and Elton P. Hsu for helpful conversations and Christopher Kelly for valuable technical assistance.

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